

PAIRS OF ORTHOGONAL COUNTABLE ORDINALS

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ABSTRACT. We characterize pairs of orthogonal countable ordinals. Two ordinals α and β are orthogonal if there are two linear orders A and B on the same set V with order types α and β respectively such that the only maps preserving both orders are the constant maps and the identity map. We prove that if α and β are two countable ordinals, with $\alpha \leq \beta$, then α and β are orthogonal if and only if either $\omega + 1 \leq \alpha$ or $\alpha = \omega$ and $\beta < \omega\beta$.

1. INTRODUCTION

The following notion has been introduced by Demetrovics, Miyakawa, Rosenberg, Simovici and Stojmenović [5]:

Two orders P and Q on the same set are *orthogonal* if their only common order preserving maps are the identity map and the constant maps.

In this paper, we say that two ordinals α and β are *orthogonal* if there exist two linear orders A and B on the same set V with order types α and β respectively such that the only maps preserving both orders are the constant maps and the identity map. Let ω be the first infinite ordinal.

We prove:

Theorem 1. *If α and β are two countable ordinals, with $\alpha \leq \beta$, then α and β are orthogonal if and only if either $\omega + 1 \leq \alpha$ or $\alpha = \omega$ and $\beta < \omega\beta$.*

The proof of Theorem 1 will be done in Section 4, using the following result.

Theorem 2. *There are 2^{\aleph_0} linear orders L of order type ω on \mathbb{N} such that L is orthogonal to the natural order on \mathbb{N} .*

This result will be given in Section 3, and follows from a simple construction which gives a bit more (see Corollary 2).

Let us say few words about the history of this notion of orthogonality and our motivation.

The notion of orthogonality originates in the theory of clones. The first examples of pairs of orthogonal finite orders were given by Demetrovics, Miyakawa, Rosenberg, Simovici and Stojmenović [5]; those orders were in fact bipartite. More examples can be found in [6]. Nozaki, Miyakawa, Pogosyan and Rosenberg [19] investigated the existence of a linear order

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orthogonal to a given finite linear order. They observed that there is always one provided that the number of elements is not equal to three and proved:

Theorem 3. [19] *The proportion $q(n)/n!$ of linear orders orthogonal to the natural order on $[n] := \{1, \dots, n\}$ goes to $e^{-2} = 0.1353\dots$ when n goes to infinity.*

Their counting argument was based on the fact that two linear orders on the same *finite* set are orthogonal if and only if they do not have a common nontrivial interval. The notion capturing the properties of intervals of a linear order was extended long ago to posets, graphs and binary structures and a decomposition theory has been developed (eg see [9], [12], [10], [7]). One of the terms in use for this notion is *autonomous set*; structures with no nontrivial autonomous subset -the building blocks in the decomposition theory- are called *prime* (or *indecomposable*). With this terminology, the above fact can be expressed by saying that two linear orders \mathcal{L} and \mathcal{M} on the same finite set V are orthogonal if and only if the binary structure $B := (V, \mathcal{L}, \mathcal{M})$, that we call a *bichain*, is prime. This leads to results relating primality and orthogonality ([22], [25]).

The notion of primality has reappeared in recent years under a quite different setting: a study of permutations motivated by the Stanley-Wilf conjecture, now settled by Marcus and Tardös [17]. This study, which developed in many papers, can be presented as follows: To a permutation σ on $[n]$ associate first the linear order \leq_σ defined by $x \leq_\sigma y$ if $\sigma(x) \leq \sigma(y)$ for the natural order on $[n]$; next associate the bichain $B_\sigma := ([n], (\leq, \leq_\sigma))$. On the set $\mathfrak{S} := \cup_{n \in \mathbb{N}} \mathfrak{S}_n$ of all permutations, set $\sigma \leq \tau$ if B_σ is embeddable into B_τ . Say that a subset \mathfrak{C} of \mathfrak{S} is *hereditary* if $\sigma \leq \tau$ and $\tau \in \mathfrak{C}$ imply $\sigma \in \mathfrak{C}$. The goal is to evaluate the growth rate of the function $\varphi_{\mathfrak{C}}$ which counts for each integer n the numbers $\varphi_{\mathfrak{C}}(n)$ of permutations σ on $[n]$ which belong to \mathfrak{C} (the Stanley-Wilf conjecture asserted that $\varphi_{\mathfrak{C}}$ is bounded by an exponential if $\mathfrak{C} \neq \mathfrak{S}$). For this purpose, simple permutations were introduced. A permutation σ is *simple* if \leq_σ and the natural order \leq on $[n]$ have no nontrivial interval in common. Arbitrary permutations being obtained by means of simple permutations, the enumeration of permutations belonging to a hereditary class of permutations can be then reduced to the enumeration of simple permutations belonging to that class. This fact was illustrated in many papers ([1], [16], see also [4] for a survey on simple permutations and [2], where the asymptotic result mentioned in Theorem 3 is rediscovered). Notably, Albert and Atkinson [1] proved that the generating series $\sum_{n \in \mathbb{N}} \varphi_{\mathfrak{C}}(n)z^n$ is algebraic provided that \mathfrak{C} contains only finitely many simple permutations. They asked for possible extensions of their result to hereditary sets containing infinitely many simple permutations.

Tools of the theory of relations provide easy ways to produce examples of hereditary sets containing infinitely many simple permutations (but not to answer the Albert-Atkinson question). Let us say that a subset \mathfrak{C} of \mathfrak{S} is an *ideal* if it is non-empty, hereditary and *up-directed*, this last condition meaning that every pair $\sigma, \sigma' \in \mathfrak{C}$ has an upper bound $\tau \in \mathfrak{C}$. Let us call *age* of a bichain B the set $\text{age}(B) := \{\sigma \in \mathfrak{S} : B_\sigma \text{ is embeddable into } B\}$. Then, a subset \mathfrak{C} of \mathfrak{S} is an ideal if and only if \mathfrak{C} is the age of some bichain B . Furthermore, an ideal \mathfrak{C} is the age of a prime bichain if and only if every permutation belonging to \mathfrak{C} is dominated by some simple permutation belonging to \mathfrak{C} (these statements, which hold in the more general context of the theory of relations, are respectively due to Fraïssé [11] and Ille [13]). Because of these results, the study of ideals leads to the study of countable prime bichains. It is then natural to ask which are the possible pairs of order types of linear orders with this property. Now, it must be noticed that in the infinite case, primality and

orthogonality no longer coincide. Thus, the next question is about pairs of orthogonal linear orders. In [23] it was proved that the chain of the rational numbers admits an orthogonal linear order of the same order type. Here we examine the case of countable well ordered chains.

2. BASIC NOTATIONS AND RESULTS

Let V be a set. A *binary relation* on V is a subset ρ of the cartesian product $V \times V$, but for convenience we write $x\rho y$ instead of $(x, y) \in \rho$. A map $f : V \rightarrow V$ *preserves* ρ if:

$$x\rho y \Rightarrow f(x)\rho f(y)$$

for all $x, y \in V$.

These two notions are enough to present our results. In order to prove them, we will need a bit more.

A *binary structure* is a pair $R := (V, (\rho_i)_{i \in I})$ where V is a set and each ρ_i is a binary relation on V . If F is a subset of V , the restriction of R to F is $R|_F := (F, ((F \times F) \cap \rho_i)_{i \in I})$. If $R := (V, (\rho_i)_{i \in I})$ and $R' := (V', (\rho'_i)_{i \in I})$ are two binary structures, a *homomorphism* of R into R' is a map $f : V \rightarrow V'$ such that the implication

$$(1) \quad x\rho_i y \Rightarrow f(x)\rho'_i f(y)$$

holds for every $x, y \in V, i \in I$. If f is one-to-one and implication (1) above is a logical equivalence, this is an *embedding*. If $R = R'$, a homomorphism is an *endomorphism*. We will denote by $R \leq R'$ the fact that there is an embedding of R into R' and by $R \leq_{fin} R'$ the fact that $R|_{(V \setminus F)} \leq R'$ for some finite subset F of V .

Let $R := (V, (\rho_i)_{i \in I})$ be a binary structure. A subset A of V is *autonomous* (other terms are interval, module and clan) in R if for all $v \notin A$ and for all $a, a' \in A$ and for all $i \in I$, the following property holds:

$$(v\rho_i a \Rightarrow v\rho_i a') \text{ and } (a\rho_i v \Rightarrow a'\rho_i v).$$

The empty set, the whole set V and the singletons in V are autonomous sets and are called *trivial*. We say that R is *prime* if it has no nontrivial autonomous set, it is *semirigid* if the identity map and the constant maps are the only endomorphisms of R and it is *embedding rigid* if the identity map is the only embedding from R to R . Finally, we say that two binary relations ρ and ρ' on a set V are *orthogonal* (or *perpendicular*) if the binary structure (V, ρ, ρ') is semirigid.

Remark 1. *Semirigidity is often defined for structures made of reflexive relations. Indeed, in that case, all constant functions are endomorphisms and, more generally, any map mapping an autonomous set A on an element $a \in A$ and leaving fixed the complement of A is an endomorphism. Thus, if R is semirigid, R must be prime, and in any case, embedding rigid.*

The binary structures we consider in this paper are made of one or two binary relations. Except in Section 3, these binary relations are orders, that is reflexive, antisymmetric and transitive relations. Our terminology on posets and chains agrees essentially with [24]. An *ordered set*, *poset* for short, is a pair $P := (V, \rho)$ where ρ is an order relation on V . Instead of ρ and $(x, y) \in \rho$ we may use the symbol \leq and write $x \leq y$. When needed, we will use other symbols like $\leq', \leq_P, \mathcal{P}$. The *dual* of P is $P^* := (V, \rho^{-1})$ where $\rho^{-1} := \{(x, y) : (y, x) \in \rho\}$.

If $P := (V, \leq)$ is a poset, the *comparability graph* of P , which we denote by $\text{Comp}(P)$, is the graph whose vertex set is V and whose edges are the pairs $\{x, y\}$ such that either $x < y$ or $y < x$. A *chain* is a poset in which the order is *linear* (or total), a *bichain* is a binary structure made of two linear orders, say \mathcal{L}, \mathcal{M} , on the same set V that we will denote $B := (V, \mathcal{L}, \mathcal{M})$ (instead of $B := (V, (\mathcal{L}, \mathcal{M}))$). If $C := (V, \mathcal{L})$ is a chain, an autonomous subset is simply an interval, that is, a subset A such that $x \leq z \leq y$ and $x, y \in A$ imply $z \in A$. If $B := (V, \mathcal{L}, \mathcal{M})$ is a bichain, an autonomous subset is a common interval of the chains (V, \mathcal{L}) and (V, \mathcal{M}) . We suppose the reader is familiar with the notions of order types of chains and of well ordered chains, alias *ordinal numbers*, as explained in [24]. In the proof of Theorem 8, an ordinal is the set of strictly smaller ordinals. We denote by ω the order type of the chain of non-negative integers (\mathbb{N}, \leq) where \leq is the natural order, and by ω^* the order type of its reverse. We use Greek letters, like α, β, \dots to denote order types. If α and β are order types of chains C and C' , then we set $\alpha \leq \beta$ if C is embeddable into C' . We denote by $\omega\alpha$ the lexicographical sum of copies of ω indexed by a chain of type α . We note that for us 0 is a limit ordinal.

We also note that the condition $\beta < \omega\beta$ in Theorem 1 could be expressed as $\text{ind}(\beta) < \omega^\omega$. Indeed, define the *right indecomposable part* of an ordinal γ as the ordinal $\text{ind}(\gamma)$ defined as follows: $\text{ind}(\gamma) := 0$ if $\gamma = 0$ and otherwise $\text{ind}(\gamma) := \delta$ where δ is the least nonzero ordinal such that $\gamma = \gamma' + \delta$ for some γ' . Then observe that:

Fact. For a nonzero ordinal γ the following properties are equivalent.

- (i) $\text{ind}(\gamma) \geq \omega^\omega$.
- (ii) $\omega\gamma = \gamma$.

Proof of the Fact. According to the Cantor Decomposition Theorem, $\gamma = \omega^{\beta_0} + \dots + \omega^{\beta_k}$ with $\beta_0 \geq \dots \geq \beta_k$. Hence, $\omega\gamma = \omega^{1+\beta_0} + \dots + \omega^{1+\beta_k}$. Since $\text{ind}(\gamma) = \omega^{\beta_k}$, $\text{ind}(\gamma) \geq \omega^\omega$ if and only if $\beta_k \geq \omega$. If $\beta_k \geq \omega$, then $\beta_i \geq \omega$ for every i , hence $1 + \beta_i = \beta_i$ for all i and therefore $\omega\gamma = \gamma$. Conversely, if $\omega\gamma = \gamma$, then since the Cantor decomposition is unique, $1 + \beta_i = \beta_i$ for every i , amounting to $\beta_k \geq \omega$. \square

Finally we recall that a map f from a chain $C := (V, \leq)$ into itself is *extensive* if $x \leq f(x)$ for every $x \in V$. We will use several times the fact that a one-to-one order preserving map on a well ordered chain is extensive.

We will use the following results.

Theorem 4. A poset $P := (V, \mathcal{P})$ is prime if and only if $\text{Comp}(P)$ is prime. Moreover, if $\text{Comp}(P)$ is prime then

- (a) the edge set of $\text{Comp}(P)$ has exactly two transitive orientations (namely \mathcal{P} and \mathcal{P}^{-1}),
- (b) if the order \mathcal{P} is the intersection of two linear orders then no other pair of linear orders yields the same intersection.

Theorem 5. Let \mathcal{L} and \mathcal{M} be two linear orders on the same set V . Then the poset $P := (V, \mathcal{L} \cap \mathcal{M})$ is prime if and only if the bichain $B := (V, \mathcal{L}, \mathcal{M})$ is prime.

Theorem 4 was obtained for finite posets in [12], and extended to the infinite in [15]. Theorem 5 was stated in [25] for a finite sets; the proof given holds without that restriction.

A characterization of pairs of infinite orthogonal linear orders is easy to state but, contrary to the finite case, it says nothing about existence. Indeed:

Theorem 6. *Let \mathcal{L} and \mathcal{M} be two linear orders on the same set V . The following properties are equivalent.*

- (i) \mathcal{L} and \mathcal{M} are orthogonal.
- (ii) The bichain $B := (V, \mathcal{L}, \mathcal{M})$ is prime and embedding rigid.
- (iii) The poset $P := (V, \mathcal{L} \cap \mathcal{M})$ is prime and has at most two embeddings: the identity map and some embedding of order 2.

Proof. (i) \Rightarrow (ii). This follows from Remark 1.

(ii) \Rightarrow (iii). Since B is prime, P is prime by Theorem 5. Let f be an embedding of P . Let $\mathcal{P} := \mathcal{L} \cap \mathcal{M}$, $\mathcal{L}_f := \{(x, y) : (f(x), f(y)) \in \mathcal{L}\}$ and let \mathcal{M}_f be defined similarly. Then f is an embedding of (V, \mathcal{L}_f) into (V, \mathcal{L}) and an embedding of (V, \mathcal{M}_f) into (V, \mathcal{M}) . Hence, $\mathcal{P} = \mathcal{L}_f \cap \mathcal{M}_f$. According to (b) of Theorem 4, $\{\mathcal{L}, \mathcal{M}\} = \{\mathcal{L}_f, \mathcal{M}_f\}$.

Case 1. $\mathcal{L} = \mathcal{L}_f$ and $\mathcal{M} = \mathcal{M}_f$. In this case, since f is an embedding of (V, \mathcal{L}_f) into (V, \mathcal{L}) , it preserves \mathcal{L} ; for the same reason, it preserves \mathcal{M} . Thus f is the identity.

Case 2. $\mathcal{L} = \mathcal{M}_f$ and $\mathcal{M} = \mathcal{L}_f$. In this case, $f \circ f$ preserves necessarily \mathcal{L} and \mathcal{M} and therefore is an embedding of B . Thus $f \circ f$ is the identity.

We check now that if f is not the identity, this is the only embedding of P distinct from the identity. Indeed, let f' be an embedding of P then $f' \circ f$ is an embedding too. If f' is distinct from the identity then $\mathcal{L} = \mathcal{M}_{f'}$ and $\mathcal{M} = \mathcal{L}_{f'}$. It follows that $f' \circ f$ preserves \mathcal{L} and \mathcal{M} . Thus $f' \circ f$ is the identity, amounting to $f' = f$.

(iii) \Rightarrow (i). Let f be an endomorphism of B . If f is not one-to-one, then the inverse image of some element under f is an interval for both chains (V, \mathcal{L}) and (V, \mathcal{M}) , hence it is an autonomous subset in B . Since P is prime, B must be prime (the easy part of Theorem 5), hence the autonomous set is V and therefore f is a constant map. If f is one-to-one this is an embedding of B , hence an embedding of P . If f is not the identity, it must have order 2. But, since f is an endomorphism of B , it preserves \mathcal{L} and since \mathcal{L} is a linear order, the orbit of an element x not fixed by f must be infinite. Hence, f is the identity. \square

3. PROOF OF THEOREM 2

Let A be a subset of \mathbb{N} . Set $\hat{A} := A \times \{1\}$ and $\mathbb{N}(A) := \mathbb{N} \cup \hat{A}$. Let $G(A) := (\mathbb{N}(A), E(A))$ be the graph (undirected, with no loop) whose vertex set is $\mathbb{N}(A)$ and edge set $E(A) := \{\{n, n+1\} : n \in \mathbb{N}\} \cup \{\{n, (n, 1)\} : n \in A\}$. For instance, $G(\emptyset)$ is the infinite one way path; the graphs $G(\{2\})$ and $G(\mathbb{N})$ are depicted in Figure 1.

Let $k \in \mathbb{Z}$, let $t_k : \mathbb{N} \rightarrow \mathbb{Z}$ be the map defined by $t_k(n) = n + k$. We call *translation* any map of that form.

Lemma 1. *If $1 \notin A$ then $G(A)$ is prime. If A is not eventually periodic, $G(A)$ is embedding rigid.*

Proof. Suppose that $1 \notin A$. Let X be a nonempty autonomous set in $G(A)$ with more than one element. Then $X \cap \mathbb{N}$ is autonomous in $G(A)_{|\mathbb{N}}$. Since a path with at least four vertices is prime, $X \cap \mathbb{N}$ must be a trivial subset of \mathbb{N} . The case $X \cap \mathbb{N} = \{n\}$ is impossible. Indeed, since X has at least two elements, it contains an element of the form $(m, 1)$. If $m \leq n$ or $n+2 \leq m$, then $\{n+1, (m, 1)\}$ is not an edge of $G(A)$, whereas $\{n, n+1\}$ is an edge. Since X is autonomous, $n+1 \in X$, which gives a contradiction. If $m = n+1$, then, since $1 \notin A$,

$m \geq 2$ and thus $n - 1$ is defined; since $\{n - 1, (m, 1)\}$ is not an edge, whereas $\{n - 1, n\}$ is an edge and X is autonomous, $n - 1 \in X$. This gives a contradiction. The case $X \cap \mathbb{N} = \emptyset$ is also impossible. Indeed, X contains two elements of the form $(m, 1), (m', 1)$. Since $\{m, (m + 1)\}$ is an edge, whereas $\{m, (m' + 1)\}$ is not, $m \in X$. It follows that $\mathbb{N} \subseteq X$. Let $m \in A$; since $\{m, (m, 1)\}$ is an edge and $\{m + 1, (m, 1)\}$ is not, $(m, 1) \in X$. Hence $X = N(A)$. Thus $G(A)$ is prime. Now, Let f be an embedding of $G(A)$ into $G(A)$. This embedding maps vertices of degree 2 to vertices of degree 2 or 3. Since those vertices belong to the infinite path $G(A)_{|\mathbb{N}}$, the map f induces an embedding of $G(A)_{|\mathbb{N} \setminus \{0\}}$ into $G(A)_{|\mathbb{N}}$. This embedding is a translation t_k for some $k \in \mathbb{N}$. The elements of $A \setminus \{0\}$ are the only elements of degree 3, from this we have $t_k(A) \subseteq A$. As is well known, if a translation sends a subset of \mathbb{N} into itself, this subset is eventually periodic. \square

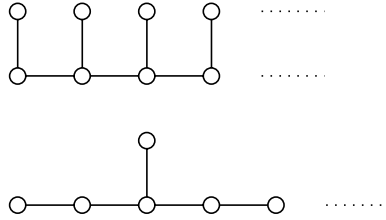


FIGURE 1.

Let A be a subset of \mathbb{N} and $P(A)$ be the poset whose vertex set is $\mathbb{N} \cup \hat{A}$, the strict order being the union of the following two sets:

$$\{(2n, 2n - 1) : n > 0\} \cup \{(2n, 2n + 1) : n \geq 0\}$$

and

$$\{(2n, (2n, 1)) : 2n \in A\} \cup \{((2n + 1, 1), 2n + 1) : 2n + 1 \in A\}.$$

Lemma 2. *The comparability graph of $P(A)$ is $G(A)$. The order on $P(A)$ is the intersection of a linear order $L(A)$ of order type ω and a linear order $L'(A)$ of order type ω^* . If $1 \notin A$ then $P(A)$ is prime; if moreover A is not eventually periodic, then $L(A)$ and $L'(A)$ are orthogonal.*

Proof. A representation of $P(A)$ into the cartesian product $\mathbb{N} \times \mathbb{N}$ is implicitly depicted in Figure 2. The first component of the cartesian product is ordered with the natural order, the second is ordered by its reverse. The two lexicographical orders on the product yield two linear extensions of $P(A)$ of order type ω and ω^* respectively. Next, apply Lemma 1: If $1 \notin A$, then $G(A)$ is prime but then $P(A)$ is prime. If A is not eventually periodic, then $G(A)$ is embedding rigid, but then, trivially, $P(A)$ is embedding rigid. If both conditions hold, then $L(A)$ and $L'(A)$ are orthogonal by Theorem 6. \square

Let A, A' be two subsets of a set E ; we recall that A is *almost included* into A' , denoted $A \subseteq_{fin} A'$, if the set $A \setminus A'$ is finite.

The second part of Lemma 1 extends as follows:

Lemma 3. *Let $A, A' \in 2^{\mathbb{N}}$. The following properties are equivalent:*

- (i) $G(A)$ is almost embeddable into $G(A')$.

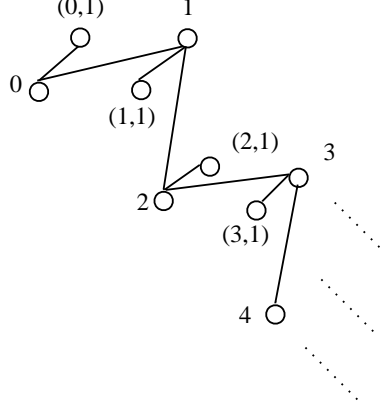


FIGURE 2.

(ii) There is some translation t_k such that $t_k(A)$ is almost included in A' .

Proof. (ii) \Rightarrow (i). Let t_k be a translation such that $t_k(A)$ is almost contained in A' . An integer $n \in \mathbb{N}$ is *bad* if either $t_k(n) \notin \mathbb{N}$ or $t_k(n) \notin A'$. The set of bad integers is finite, hence the set F of integers dominated by some bad integer is finite too. Let $X := (\mathbb{N} \cup \hat{A}) \setminus (F \cup \hat{F})$ and $\bar{t}_k : X \rightarrow \mathbb{N} \cup \hat{A}'$ defined by $\bar{t}_k(n) = t_k(n)$ if $n \in \mathbb{N} \setminus F$ and $\bar{t}_k(n, 1) = (t_k(n), 1)$ if $n \in A \setminus F$. This defines an embedding from $G(A)_{|X}$ into $G(A')$.

(i) \Rightarrow (ii). Let f be an embedding of a restriction of $G(A)$ to a cofinite set of $\mathbb{N}(A)$ into $G_{A'}$. This embedding maps vertices of degree 2 to vertices of degree 2 or 3. Since these vertices belong to the infinite path $G(A')_{|\mathbb{N}}$, the map f induces an embedding of $G(A)_{|Y}$, where Y is an infinite final segment of \mathbb{N} , into $G(A')_{|\mathbb{N}}$. Thus, such a map is the restriction of some t_k where $k \in \mathbb{Z}$. Since the elements of A and A' (except at most one) are the only elements of degree 3, f maps almost all elements of A into A' , hence $t_k(A)$ is almost included in A' . \square

Lemma 4. *There is a family \mathcal{A} of 2^{\aleph_0} subsets of \mathbb{N} such that for every pair A, A' of distinct elements of \mathcal{A} , no translate of A is almost included in A' .*

Proof. We present two proofs. For the first one, start with $X := \{x_n : n \in \mathbb{N}\}$ where $x_0 = 0$ and $x_{n+1} = x_n + n$ (one just need the gaps increasing). Now, let \mathcal{A} be an almost disjoint family of 2^{\aleph_0} infinite subsets of X . For any $A \in \mathcal{A}$, and $n > 0$, $A + n$ is almost disjoint from X , and thus almost disjoint from any other A' .

The second proof makes use of Sturmian words ([14], [3]). We identify subsets of \mathbb{N} with their characteristic functions, that is binary words. Hence, translating a set corresponds to shifting the word (note that in this correspondence, if u and v are two infinite binary words, we have $u \leq v$ iff $u^{-1}(1) \subseteq v^{-1}(1)$; however some translate of $u^{-1}(1)$ can be contained into $v^{-1}(1)$ whereas no iterated shift of u is almost contained in v , an observation leading to Problem 1 below).

Let $\alpha \in (0, 1) \setminus \mathbb{Q}$. Let X_α be the set of Sturmian words whose slope is α (here the *slope* is the frequency of the letter 1). The set X_α is a minimal uncountable subshift which is *balanced*: for any two finite binary words u and v that appear as factors of elements of X_α , if u and v have the same length, then $|u|_1 - |v|_1 \leq 1$ (where $|u|_1$ denotes the number of occurrences of the letter 1 in u).

Let us define the following equivalence relation on X_α : $x \sim y$ if there exists two integers p and q such that $S^p(x) = S^q(y)$ (S denotes the shift map and S^p its p iterates, i.e S is the map from $2^\mathbb{N}$ to $2^\mathbb{N}$ defined by $S(x)_i := x_{i+1}$ for $i \in \mathbb{N}$). Since each class is countable, the quotient X_α / \sim is uncountable: Let $\mathcal{A} \subseteq X_\alpha$ be a system of representatives of X_α / \sim .

Now, let x and y be two elements of \mathcal{A} such that there exists an integer n such that $S^n(y)$ is almost included in x , let us prove that $x = y$.

There exists an integer k such that for any index i , $S^k(x)_i \geq S^{n+k}(y)_i$. If $S^k(x) = S^{n+k}(y)$, then $x = y$. Otherwise, there exists an index i such that $S^k(x)_i = 1$ and $S^{n+k}(y)_i = 0$. Since $S^k(x)$ and $S^{n+k}(y)_i$ are elements of X_α which is balanced, $S^k(x)_i = S^{n+k}(y)_i$ for any $j \neq i$, in particular $S^{k+i+1}(x) = S^{n+k+i+1}(y)$ and again $x = y$. \square

We thank Thierry Monteil [18] for providing the second proof. We thank the referee for providing an alternative proof using words obtained by irrational rotations. For a link between irrational rotations and Sturm words, see Chapter 6 of [8].

Problem 1. *Let us recall that a subset X of $2^\mathbb{N}$ is shift-invariant if $S(X) \subseteq X$, where $S(X) := \{S(x) : x \in X\}$. It is minimal if it is non-empty, compact, shift-invariant and no proper subset has the same properties (cf [3]). For example the set X_α of Sturmian words with slope α is minimal. Is it true that every infinite minimal set X contains a subset X' of cardinality 2^{\aleph_0} such that for every distinct $u, u' \in X'$, no translate of $u^{-1}(1)$ is almost contained in $u'^{-1}(1)$?*

Corollary 1. *There is a family \mathcal{A} of subsets of $\mathbb{N} \setminus \{0, 1\}$ indexed by the positive reals such that $G(A)$ is prime and embedding rigid for every $A \in \mathcal{A}$, and furthermore $G(A) \not\leq_{fin} G(A')$ for all distinct $A, A' \in \mathcal{A}$.*

Proof. Apply Lemma 4. \square

Corollary 2. *There is a family \mathcal{L} of 2^{\aleph_0} linear orders on \mathbb{N} of order type ω which are orthogonal to ω . Furthermore, $(\mathbb{N}, \leq, \leq_L) \not\leq_{fin} (\mathbb{N}, \leq, \leq_{L'})$ for every distinct $L, L' \in \mathcal{L}$.*

Proof. Select \mathcal{A} as in Corollary 1 and apply Lemma 2. To A associate the prime poset $P(A)$ then the unique bichain $B(A) := (\mathbb{N} \cup \hat{A}, L(A), L'(A))$ where $L(A)$, $L'(A)$ have order type ω and ω^* respectively and such that the intersection $L(A) \cap L'(A)$ is the order of $P(A)$. The bichain $B(A)$ is semi-rigid and $B(A) \not\leq_{fin} B(A')$ for $A \neq A'$. Replace each $B(A)$ by the bichain $B^*(A) := (\mathbb{N} \cup \hat{A}, (L(A), L'^*(A)))$ where $L'^*(A)$ is the dual of $L'(A)$. These bichains enjoy the same properties as the $B(A)$'s. The components of these bichains being chains of order type ω , we may suppose via a bijective map that their common domain is \mathbb{N} and the first order is the natural order. This yields the above collection. \square

4. PROOF OF THEOREM 1

Theorem 7. *An ordinal α is orthogonal to ω if and only if α is countably infinite and $\omega\alpha > \alpha$.*

Trivially, an ordinal α orthogonal to ω must be countably infinite. The fact that $\omega\alpha > \alpha$ is a consequence of the next lemma.

Lemma 5. *If α is a countably infinite order type (not necessarily an ordinal) and $\omega\alpha \leq \alpha$, then the bichain $(\mathbb{N}, \leq, \leq_L)$, where $L := (\mathbb{N}, \leq_L)$ is a chain of order type α , is not embedding rigid.*

For the proof of Lemma 5, we use the following result, which is essentially Lemma 3.4.1 of [21].

Lemma 6. *Let α be a countably infinite order type, $L_1 := (\mathbb{N}, \leq_{L_1})$ and $L_2 := (\mathbb{N}, \leq_{L_2})$ be two chains of order type α and $\omega\alpha$ respectively. Then there is an embedding of $(\mathbb{N}, \leq, \leq_{L_1})$ into $(\mathbb{N}, \leq, \leq_{L_2})$.*

Proof of Lemma 5. Let $L_1 := (\mathbb{N}, \leq_{L_1})$ be a chain with order type α . Since $\omega\alpha \leq \alpha$, there is a subset X of \mathbb{N} , and in fact a proper subset, such that $L_1 \upharpoonright X$ has order type $\omega\alpha$. Let h be the unique order isomorphism from (\mathbb{N}, \leq) onto (X, \leq_{L_1}) and let $L_2 := (\mathbb{N}, \leq_{L_2})$ where $x \leq_{L_2} y$ amounts to $h(x) \leq_{L_1} h(y)$. Clearly, L_2 has order type $\omega\alpha$ and h is an embedding of $(\mathbb{N}, \leq, \leq_{L_2})$ into $(\mathbb{N}, \leq, \leq_{L_1})$. According to Lemma 6, there is an embedding, say f , of $(\mathbb{N}, \leq, \leq_{L_1})$ into $(\mathbb{N}, \leq, \leq_{L_2})$. The map $h \circ f$ is an embedding of $(\mathbb{N}, \leq, \leq_{L_1})$ into $(\mathbb{N}, \leq, \leq_{L_1})$. Since $X \neq \mathbb{N}$, this map is not surjective, hence this is not the identity and thus $(\mathbb{N}, \leq, \leq_{L_1})$ is not embedding rigid. \square

It remains to prove that if α is countable and $\text{ind}(\alpha) < \omega^\omega$, then α is orthogonal to ω . Note that $0 \neq \text{ind}(\alpha) < \omega^\omega$ amounts to $\text{ind}(\alpha) = \omega^n$ for some integer n . The proof will proceed by induction on n after some necessary lemmas.

Lemma 7. *Let $n < \omega$ and f be an embedding from a chain C of order type ω^{n+1} into itself which is not the identity. Let $(C_\alpha)_{\alpha < \omega^n}$ be the decomposition of C into intervals of order type ω . Then there are $\alpha \leq \beta$ such that f is not the identity on C_α and $f(C_\alpha) \setminus C_\beta$ is finite. Furthermore, $f(C_\alpha) \subseteq C_\alpha$ if $\alpha = \beta$.*

Proof. We mention at first that the existence of α and β such that $f(C_\alpha) \setminus C_\beta$ is finite implies $\alpha \leq \beta$ and $f(C_\alpha) \subseteq C_\alpha$ if $\alpha = \beta$. Indeed, since C is well ordered, $f(x) \geq x$ for every $x \in C$, hence $C_{\alpha'} \cap f(C_\alpha) = \emptyset$ for every $\alpha' < \alpha$. Now the proof of the lemma goes by induction on n . If $n = 0$ then $C = C_0$. Set $\alpha = \beta = 0$. Since $f(C_0) \subseteq C_0$, we have $f(C_0) \setminus C_0 = \emptyset$, thus this set is finite, as required, and we are done.

Let $n \geq 1$ and suppose that the property holds for $n' < n$. Let $(A_k)_{k < \omega}$ be the decomposition of C into intervals of order type ω^n .

Claim 1. *There is an embedding $\phi : \omega \rightarrow \omega$ such that for each $k < \omega$, $f(A_k) \setminus A_{\phi(k)}$ has order type $< \omega^n$.*

Proof of Claim 1. Let $k < \omega$. Set $i(k) := \{l < \omega : f(A_k) \cap A_l \neq \emptyset\}$. This set is finite and nonempty. Set $\phi(k) := \max(i(k))$. As it is easy to see, the map ϕ is an embedding from ω into ω . \square

Claim 2. *There is some element $a \in A_0$ such that the decomposition of $A'_0 := \{x : a \leq x\} \cap A_0$ into intervals of order type ω is induced by the decomposition of C into intervals of order type ω and $f(A'_0) \subsetneq A''_0 := A_{\phi(0)}$.*

Proof of Claim 2. According to Claim 1, $f(A_0) \setminus A''_0$ has order type $< \omega^n$. Since $f(A_0)$ has order type ω^n , there is some $x \in A_0$ such that $f(\{y : x \leq y\} \cap A_0) \subseteq A''_0$. Pick $a > x$ in A_0 such that the decomposition of $A'_0 := \{x : a \leq x\} \cap A_0$ into intervals of order type ω is induced by the decomposition of C into intervals of order type ω . \square

With these two claims the proof of the lemma goes as follows. First, with no loss of generality, we may suppose that f is not the identity on A_0 . Otherwise, let k_0 be the least integer k such that f is not the identity on A_{k_0} . Let $C' := C \setminus \cup_{k < k_0} A_k$ and $(A'_k)_{k < \omega}$ be

the decomposition of C' into intervals of order type ω^n . Then $A'_0 = A_{k_0}$ and f induces an embedding f' of C' into itself which is not the identity on A'_0 . Thus, we may replace C and f by C' and f' .

With our supposition, set $C' := C_{\upharpoonright A'_0}$. Let h be an order isomorphism of $C_{\upharpoonright A'_0}$ onto C' and $f' := h \circ f$. Then f' is not the identity on A'_0 . Induction applied to C' and f' yields some $\alpha' \leq \beta' < \omega^{n-1}$ such that f is not the identity on $C'_{\alpha'}$ and $f'(C'_{\alpha'}) \setminus C'_{\beta'}$ is finite. Let α and β be such that $C'_{\alpha'} \subseteq C_\alpha$ and $C_\beta = h^{-1}(C'_{\beta'})$. \square

Lemma 8. *For every $n < \omega$ the ordinals ω and ω^{n+1} are orthogonal.*

Proof. If $n = 0$, the result follows from Theorem 2. We suppose that $n \geq 1$.

Claim 3. *There is a family of $(X_\alpha)_{\alpha < \omega^n}$ of infinite subsets of \mathbb{N} such that none contains a nontrivial interval and $0 \notin X_0$.*

Proof of Claim 3. Let $(X_\alpha)_{\alpha < \omega^n}$ be a partition of \mathbb{N} into infinitely many pairwise infinite subsets of \mathbb{N} , where X_0 is the set of odd integers. \square

Claim 4. *There is a chain $C := (\mathbb{N}, \leq_C)$ of order type ω^{n+1} such that $(X_\alpha)_{\alpha < \omega^{n+1}}$ is the decomposition of C into intervals of order type ω . If C is such a chain then the bichain $(\mathbb{N}, \leq, \leq_C)$ is prime. Furthermore, if for each $\alpha < \omega^n$, $B_\alpha := (X_\alpha, \leq_{\upharpoonright X_\alpha}, \leq_{C \upharpoonright X_\alpha})$ is embedding rigid and for $\alpha < \beta$, $B_\alpha \not\leq_{fin} B_\beta$ then $(\mathbb{N}, \leq, \leq_C)$ is semirigid.*

Proof of Claim 4. The existence of the chain C is pretty obvious: on each set X_α choose a linear order of order type ω , and for $\alpha < \beta$ put every element of X_α before every element of X_β . Suppose for a contradiction that $(\mathbb{N}, \leq, \leq_C)$ is not prime. Let I be a nontrivial autonomous set, that is an interval for \leq and \leq_C . Due to our condition on the family $(X_\alpha)_{\alpha < \omega^n}$, no X_α can contain I . Thus there are α, β with $\alpha \neq \beta$ such that X_α and X_β meet I . We may suppose that $\alpha < \beta$. In such a case I contains infinitely many elements of X_α . This implies that I is a final segment of (\mathbb{N}, \leq) . For each $n \in \mathbb{N}$, let $\phi(n)$ be such that $n \in X_{\phi(n)}$. For each $i \notin I$, $X_{\phi(i)} \cap I$ is infinite. It follows that $\mathbb{N} \setminus I$ cannot contain two distinct elements, hence $\mathbb{N} \setminus I = \{0\}$. Since $0 \notin X_0$, 0 is not the least element of C , hence $I = \mathbb{N} \setminus \{0\}$ is not an interval of C , a contradiction.

Finally, we show that $(\mathbb{N}, \leq, \leq_C)$ is embedding rigid. Suppose for a contradiction that there is a proper embedding, say f . Since f is an embedding of C , Lemma 7 ensures that there are $\alpha \leq \beta < \omega^n$ such that:

- (1) f is not the identity on X_α .
- (2) $f(X_\alpha)$ is contained in X_α if $\beta = \alpha$ and almost contained in X_β otherwise.

Since B_α is embedding rigid, $\alpha \neq \beta$. Set $X'_\alpha = X_\alpha \cap f^{-1}(X_\beta)$. Then $f_{\upharpoonright X'_\alpha}$ is an embedding of $B_{\alpha \upharpoonright X'_\alpha}$ into B_β . Since $X_\alpha \setminus X'_\alpha$ is finite $B_\alpha \leq_{fin} B_\beta$, a contradiction. \square

According to Corollary 2 in Section 3, there is a family $(B_\alpha)_{\alpha < \omega^n}$ satisfying the conditions of Claim 4. Hence, ω and ω^{n+1} are orthogonal. \square

Let α be an ordinal and let $0 < n < \omega$. For $i < n$ define:

$$\bar{i}(\text{mod } n) := \{\beta < \alpha : \beta = \gamma + i + kn \text{ for some limit ordinal } \gamma \text{ and } k < \omega\}.$$

Lemma 9. *Let α be an ordinal, $0 < n < \omega$ and $i < n$.*

(a) *If f is an embedding of α into itself which is the identity on $\bar{i}(\text{mod } n)$, then f is the identity map on α .*

(b) If α is a limit ordinal and I is an interval of α such that $\bar{i}(\text{mod } n) \subseteq I$, then $\alpha \setminus \{0, \dots, i-1\} \subseteq I$.

Proof. (a) We suppose that f is not the identity and we argue for a contradiction. Let $\beta \in \alpha$ such that $f(\beta) \neq \beta$. Set $\beta_0 := \beta$ and $\beta_{m+1} := f(\beta_m)$ for $m \in \mathbb{N}$. Since α is an ordinal and f is an embedding, we have $\beta_0 < \beta_1 < \dots < \beta_m < \dots$. Hence, there is some $i' \in \bar{i}(\text{mod } n)$ and some $m \in \mathbb{N}$ such that $\beta_m < i' < \beta_{m+1}$. Since f is order preserving $\beta_{m+1} = f(\beta_m) < f(i')$. Hence, f is not the identity on $\bar{i}(\text{mod } n)$.

(b) Observe that $\alpha \setminus \{0, \dots, i-1\}$ is the least interval containing $\bar{i}(\text{mod } n)$. \square

Lemma 10. Let α and β be two infinite ordinals with α orthogonal to β and γ be an ordinal such that $|\gamma| \leq |\alpha|$, then

- (i) α is orthogonal to $\gamma + \beta$.
- (ii) α is orthogonal to $\beta + \gamma + 1$.

Proof. Let α' be a limit ordinal and $n < \omega$ such that $\alpha = \alpha' + n$. Let $C := (V, \leq)$ be a chain with order type α and let h be an order isomorphism from α onto C . Let V' be the image of α' by h and $U := V \setminus V'$ (hence $|U| = n$). Let $\{V'_0, V'_1\}$ be the partition of V' , which is the image by h of the partition of α' into $\bar{0}(\text{mod } 2)$ and $\bar{1}(\text{mod } 2)$. Let $X \subseteq V'_1$ be such that X is an initial interval of $C \upharpoonright V'_1$ and $|X| = |\gamma|$ (this is possible since $|\gamma| \leq |\alpha|$) and set $Y := V \setminus X$. Observe that $C \upharpoonright Y$ has order type α .

(i) We notice that if $\gamma < \omega$, then $\gamma + \beta$ is isomorphic to β and hence $\gamma + \beta$ is orthogonal to α . So we may assume that γ is infinite. We define $C' := (V, \leq_{C'})$ such that $C' \upharpoonright X$ is an initial interval of C' and has order type γ , Y is a final interval of C' with order type β and $C' \upharpoonright Y$ is orthogonal to $C \upharpoonright Y$.

By construction, C' has order type $\gamma + \beta$. We claim that C' is orthogonal to C . We prove first that there is no proper common interval. Let I be a common interval of C and C' such that $|I| \geq 2$. We prove that $I = V$. Let $J := I \cap Y$. Then J is an interval of $C \upharpoonright Y$ and of $C' \upharpoonright Y$. Since $C \upharpoonright Y$ and $C' \upharpoonright Y$ are orthogonal, either J is empty, or $J = \{y\}$ for some $y \in Y$ or $J = Y$. In the first case, $I \subseteq X$, but since, in C , there is an element of Y between any two elements of X , I cannot be an interval of C . Thus this case is impossible. In the second case, there is some $x \in I \cap X$. Since β is a limit ordinal, the interval $[x, y]$ of C' contains infinitely many elements of X , hence $I \cap X$ is infinite. Since $X \subseteq V'_1$, I contains infinitely many elements of Y , a contradiction. Thus, we have $J = Y$. Since V' is the unique interval of C containing Y , we have $V' \subseteq I$ and since $Y \subseteq I$ we get $I = V$, as required.

Let f be an injective order preserving map common to C and C' . Then $f(Y) \subseteq Y$ (indeed, since C' is well ordered, f is extensive on C' ; since Y is a final segment of C' , it follows that $f(Y) \subseteq Y$). Since $C' \upharpoonright Y$ is orthogonal to $C \upharpoonright Y$, it follows that f is the identity map on Y . From that, it follows that $f(X) \subseteq X$ and also $f(V') \subseteq V'$. From Lemma 9 (i) applied to α' and $f|_{V'}$ it follows that f is the identity map on V' , thus f is the identity map and we are done.

(ii) We consider two cases:

a) γ is an infinite ordinal.

The proof follows the same lines as the proof of (i). Let u be the least element of C and v be the least element of X in C , hence v is the successor of u in C . We set $Y' := Y \setminus \{u\}$ and

$X' := X \cup \{u\}$. We define $C' := (V, \leq_{C'})$ such that $C' \restriction Y'$ is an initial interval of C' , has order type β , and is orthogonal to $C \restriction Y'$, and X' is a final interval of C' with order type $\gamma + 1$ and u and v are its least and largest element.

By construction, C' has order type $\beta + \gamma + 1$. We claim that C' is orthogonal to C . As before, we prove first that C and C' have no proper common interval. Let I be a common interval to C and C' such that $|I| \geq 2$. We prove that $I = V$.

We set $J := I \cap Y'$. Then J is an interval of $C \restriction Y'$ and of $C' \restriction Y'$. Since $C \restriction Y'$ and $C' \restriction Y'$ are orthogonal, either J is empty, or $J = \{y'\}$ for some $y' \in Y'$ or $J = Y'$. In the first case, $I \subseteq X'$, but since, in C , there is an element of Y between any two elements of X , I cannot be an interval of $C \restriction X$. Hence $I = \{u, v\}$, but since u and v are the extreme elements of the infinite chain $C' \restriction X'$, this is impossible. In the second case, there is some $x' \in I \cap X'$. The interval $[y', x']$ of C' must contain the least element of $C' \restriction X'$, that is u ; having more than one element, I must contain v , hence it contains X' , thus $I \cap X$ is infinite. Since $X \subseteq X'_1$, I contains infinitely many elements of Y , a contradiction. Thus, we have $J = Y'$. Since $V \setminus \{u, v\}$ is the smallest interval of C containing Y' and $Y' \subseteq I$, we have $V \setminus \{u, v\} \subseteq I$. Since $V \setminus \{v\}$ is not an interval of C and $V \setminus \{u\}$ is not an interval of C' it follows that $I = V$ as required.

Let f be an injective order preserving map for both C and C' . Then $f(v) = v$ (this is because f is extensive on C' and v is the largest element of C'). Also, $f(u) = u$ (this is because $f(v) = v$ and v is the successor of u in C). Since every element of Y' is below u in C' we infer that $f(Y') \subseteq Y'$. Since $C \restriction Y'$ and $C' \restriction Y'$ are orthogonal, f is the identity on Y' . Hence, $f(X') \subseteq X'$ and, from the fact that f fixes the first two element of C , it follows that $f(V') \subseteq V'$. From Lemma 9 (i) applied to α' and $f_{\restriction V'}$ it follows that f is the identity map on V' , thus f is the identity and we are done.

b) γ is finite.

In this case, it suffices to prove that the conclusion holds if $\gamma = 0$. Indeed, a straightforward induction yields the general conclusion.

Let β' be a limit ordinal and $m < \omega$ such that $\beta = \beta' + m$. Let $C' := (V, \leq')$ be a chain of type β with C' orthogonal to C . Let U' be the final segment of C' such that $|U'| = m$. Since U' is finite, $V' \setminus U'$ contains two elements a and b which are consecutive in C . Add to V an extra element v . Let C'_1 be the chain obtained by putting v as its last element and C_1 be obtained by inserting v between a and b . Trivially, C'_1 has order type $\beta + 1$; since $\{a, b\} \subseteq V'$, C_1 has order type α . We claim that C_1 and C'_1 are orthogonal.

We prove first that there is no proper common interval. Let I be a common interval to C_1 and C'_1 such that $|I| \geq 2$. We prove that $I = V \cup \{v\}$. Let $J := I \cap V$. Then J is an interval of $C_1 \restriction V$ and of $C'_1 \restriction V$. Since $C_1 \restriction V = C$ and $C'_1 \restriction V = C'$, these chains are orthogonal, hence either J is empty, or $J = \{w\}$ for some $w \in V$ or $J = V$. The first case is impossible since $|I| \geq 2$. In the second case, $v \in I$. But then I , as an interval of C'_1 , contains a or b . But then, as an interval of C'_1 , I contains either the interval $[a, v]$ or the interval $[b, v]$ of C'_1 . Both intervals are infinite, hence I is infinite, contradicting the fact that J is a singleton. Thus this case is impossible. It follows that $J = V$. Since in C_1 , v is between two elements of I , $v \in I$, thus $I = V \cup \{v\}$ as required.

Let f be an injective order preserving map common to C_1 and C'_1 . Then $f(v) = v$ (this is because f is extensive on C'_1 and v is the largest element of C'_1). Thus f induces an order

preserving map of C and C' . Since these chains are perpendicular, f is the identity on V . It follows that f is the identity on $V \cup \{v\}$ as required. The proof of the lemma is now complete. \square

Proof of Theorem 7.

Let α be a countable ordinal such that $\text{ind}(\alpha) < \omega^\omega$. We have $\text{ind}(\alpha) = \omega^n$ with $n < \omega$. If $n = 0$ then $\alpha = \omega + \gamma + 1$. According to Theorem 2, ω is orthogonal to itself. Hence from (ii) of Lemma 10, ω is orthogonal to $\omega + \gamma + 1 = \alpha$. If $n > 0$ apply Lemma 8 and (i) of Lemma 10. \square

Theorem 8. *If α and β are two countable ordinals, with $\omega + 1 \leq \alpha \leq \beta$, then α and β are orthogonal.*

Proof. The case $\alpha = \beta = \omega + 1$ follows from the fact that ω is orthogonal to ω and Lemma 10 (ii) applied twice. Thus we may suppose $\beta \geq \omega + 2$.

Let α' and β' be such that $\alpha = \omega + 1 + \alpha'$ and $\beta = \omega + 1 + \beta'$. Let V be a countably infinite set disjoint from $\omega + 1$ and let \leq_A and \leq_B be two orthogonal linear orders on V of order type ω . Let W_α and W_β be two disjoint subsets of the set of odd integers with cardinality $|\alpha'|$ and $|\beta'|$ respectively. Set $V' := V \cup W_\alpha \cup W_\beta \cup \{\omega\}$. We define two orthogonal linear orders \leq_α and \leq_β on V' with order type α and β respectively, as follows

Let f_α be the order isomorphism from $\omega \setminus W_\beta$ onto (V, \leq_A) , let g_α be any bijection from $\alpha \setminus (\omega + 1)$ onto W_α and 1_β be the identity map on $W_\beta \cup \{\omega\}$. Then $h_\alpha := f_\alpha \cup g_\alpha \cup 1_\beta$ is a bijective map from α onto V' . The order \leq_α is the image by h_α of the order on α , thus has the same order type. We define \leq_β similarly with f_β, g_β and 1_α .

To show that \leq_α and \leq_β are orthogonal, we first show that they have no nontrivial common interval. So suppose for a contradiction that there is some nontrivial common interval I . If I were to meet V in at least 2 places, then since $I \cap V$ is a common interval of \leq_A and \leq_B , which are orthogonal, we would have $V \subseteq I$; since W_β is included in the least interval of ω containing $h_\alpha^{-1}(I)$ we would have $W_\beta \subseteq I$ and, similarly, $W_\alpha \subseteq I$ and this would imply $\omega \in I$, hence $I = V'$, contradicting the non triviality of I . If I were to meet $V \cup \{\omega\}$ in two places, then it would also have to meet V in two places, and we have just shown that this cannot happen. So I must meet either W_α or W_β . If it meets W_α then it must also meet V since it is a nontrivial interval of \leq_β and W_α does not contain any two successive numbers. But then since it meets both V and W_α and is an interval of \leq_α it must contain at least two elements of V , which is the desired contradiction. The case that I meets W_β leads to a contradiction in a symmetric way.

Next, we must show that there is no nontrivial embedding f of $(V', \leq_\alpha, \leq_\beta)$ into itself. Suppose for a contradiction that there is such an embedding. It must preserve the final segment $\{\omega\} \cup W_\alpha$ of \leq_α and the final segment $\{\omega\} \cup W_\beta$ of \leq_β , and so must fix ω . Thus since it respects \leq_α it must map V into $V \cup W_\beta$ and since it respects \leq_β it must map V into $V \cup W_\alpha$: putting these two facts together, it must map V into itself. But then since \leq_A and \leq_B are orthogonal, the restriction of f to V must be the identity map. But then since f respects \leq_β the restriction of f to W_α must be the identity map. Similarly, the restriction of f to W_β must be the identity map. Thus f itself must be the identity map. \square

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